# A Ring-Theorist's Description of Fedosov Quantization

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ABSTRACT We present a formal, algebraic treatment of Fedosov's argument that the coordinate algebra of a symplectic manifold has a deformation quantization. His remarkable formulas are established in the context of affine symplectic algebras.
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One of the standard techniques for reducing a problem about a noncommutative algebra $B$ to one about commutative rings is to find a filtration $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ with $\bigcup B_i = B$ and $B_iB_j \subseteq B_{i+j}$ so that the associated graded algebra is commutative. There is a shadow of noncommutativity which can be recovered inside $grB$: if $\overline{a} \in gr_mB$ and $\overline{b} \in gr_nB$, pull back $\overline{a}$ and $\overline{b}$ to $a$ and $b$ in $B_m$ and $B_n$ respectively. Then the image of the commutator $[a, b]$ in $gr_{m+n-1}(B)$ depends only on $\overline{a}$ and $\overline{b}$. This new binary operation in $grB$ is an example of a Poisson bracket.

In general, a (commutative) Poisson algebra $A$ over the scalar field $k$ is a commutative $k$-algebra which is, at the same time, a Lie algebra under a second operation $\{*, *\} : A \times A \to A$ such that the "product" or "Liebniz" rule holds,

$$\{ab, c\} = a\{b, c\} + \{a, c\}b \quad \text{for all } a, b, c \in A.$$

It is well known that the example $grB$ described above is a Poisson algebra. It can be generalized via the Rees construction to a special case of the following artifact. Suppose the algebra $C$ has a central non-zero-divisor $h$ such that $C/hC$ is a commutative algebra. Then $C/hC$ is a Poisson algebra under the bracket $\{\overline{c_1}, \overline{c_2}\} = \frac{1}{h}\overline{[c_1, c_2]}$. (Here the overbar denotes the canonical homomorphism.)

A Poisson algebra has a "naive quantization" if the process can be reversed; given a Poisson algebra $A$, there is an algebra $C$ as we've just described such that $C/hC$ is isomorphic to $A$. One can ask for much more. The Poisson algebra $A$ has a "deformation" or "$*$-product" quantization provided there is a formal deformation of the multiplication in $A$,

$$(a, b) \mapsto ab + \{a, b\}h + \sum_{t=2}^{\infty} \mu_t(a, b)h^t$$

which satisfies the traditional associative identity and $\mu_t(a,b) = (-1)^t \mu_t(b,a)$.

One of the earliest profound results about formal quantization of Poisson algebras is the theorem of DeWilde and Lecomte ([DL]) asserting that if $M$ is a symplectic manifold then $C^\infty(M)$ has a $*$-product quantization. In the 1990s, Fedosov ([Fe]) produced a new proof which he described as a "simple geometric construction". The purpose of the article which follows is to convince the reader that Fedosov's construction is fundamentally algebraic. Whether it is actually simple we leave to others to decide. Of course, the the theorem in question has been superseded rather spectacularly by Kontsevich's description of deformations for arbitrary Poisson manifolds. The author is not aware of any clear relationship between this new construction and Fedesov's.

Our context for interpreting Fedosov's techniques is the study of commutative affine algebras over a field $k$ of characteristic zero, which are symplectic algebras. In earlier work, Loose ([L]), Farkas, and Letzter ([FL]) discussed possible definitions of a symplectic algebra which either generalize or are analogues of $C^\infty(M)$. In the "smooth" case that $A$ is a regular affine domain, all of the proposed approaches agree. Since we limit ourselves to these algebas, we can define $A$ to be symplectic provided it is also a Poisson algebra whose Poisson structure produces sufficiently many derivations. In detail, for each $a \in A$ let $ham\, a$ denote the $k$-linear derivation of $A$ which sends $b$ to $\{a,b\}$. Then $A$ is *symplectic* when these derivations generate all $k$-linear derivations as an $A$-module. (We use the shorthand $Der\, A = A \cdot Ham\, A$.) Following Fedosov, we shall prove that every regular affine symplectic domain has a naive quantization. (In fact, the noncommutative algebra which is exhibited is a deformation quantization, which the interested reader will be able to verify with ease.)

In very broad strokes, the naive quantization arises as the infinitesimal fixed ring of a universal enveloping algebra. The Poisson bracket on $A$ induces an $A$-bilinear alternating form $\omega$ on $Der\, A$, the module of $k$-derivations from $A$ to itself, by extending $\omega(ham\, a, ham\, b) = \{a,b\}$. Form the universal enveloping algebra $U$ with coefficients in $A$ based on the Lie algebra $Ah \oplus Der\, A$ where $h$ is central and $[X,Y] = h\omega(X,Y)$ for derivations $X$ and $Y$. This construction appears frequently in the literature as an appropriate generalization of the Weyl algebra, cf. [B],[NR]. Roughly speaking, there is a derivation $D$ of $U$ for which $\{x \in U | D(x) = 0\}$ turns out to be the quantization of $A$. It is, perhaps, worth noting that nothing so elaborate is required if we restrict attention to graded symplectic algebras; they all appear as the associated graded algebra for the ring of differential operators on some regular domain ([F1]).

The mathematical reality is that the enveloping algebra is too small on two counts. First, it must be completed and, secondly, we throw all differential $n$-forms with coefficients into this larger algebra. The derivation $D$ can then be identified with a connection as generalized by Fedosov. Most of our work is devoted to setting up an algebraic apparatus for this ring so that some marvelous formulas of Fedosov can be invoked. In particular, the Poisson bracket for a symplectic algebra induces a duality between $Der\, A$ and the module of

Kähler differentials on $A$. Fedosov extends the duality to a "homotopy inverse" relation of differentials on the DGA of $n$-forms just described. This can be done with "local" computations in the geometric context, which are not available algebraically. Instead, we have to jump back and forth between $U$ and the copy of its symmetrization which injects into $U$. (The reader is not expected to know any symplectic geometry in order to follow this paper. We hope the open-minded reader will recognize the potential of algebra to clarify some mathematical ideas and to open the door to abstractions which allow other problems to be resolved.)

This paper was submitted in the early months of 1995 to a collection honoring the memory of S. A. Amitsur. Shimshon was especially encouraging in his support of my project to develop symplectic themes algebraically. Unfortunately, inappropriate action of an editor led to the demise of the article. It was judged moot upon the final acceptance of [D]. I was surprised and delighted when Professor Sternheimer suggested in November 1999 that the manuscript be resurrected in its present form. This version contains the proofs referred to or abbreviated in the published paper of the same title.

# 1 The rings

For the duration of this paper $k$ is a field of characteristic zero and $A$ is a commutative affine domain which is regular and symplectic. The regularity assumption will appear in the following form: the $A$-module of $k$-linear Kähler differentials, $\Omega^1(A)$, is a finitely generated projective $A$-module. We will find it useful to exhibit projective bases explicitly. There is a standard duality (cf. [MR]) between $\Omega^1(A)$ and $Der\,A$, the module of $k$-linear derivations from $A$ to itself. If $Z \in Der\,A$ and $u\,dv \in \Omega^1(A)$ then the pairing is given by $\langle Z, u\,dv\rangle = uZ(v)$. We may choose a projective basis for $\Omega^1(A)$ of the form

$$\{(da_1, \langle X_1, -\rangle), (da_2, \langle X_2, -\rangle), \ldots, (da_n, \langle X_n, -\rangle)\}$$

where $a_1, \ldots, a_n \in A$ and $X_1, \ldots, X_n \in Der\,A$. In this case we have what we have termed an *elite* basis for $Der\,A$,

$$Y = \sum_j Y(a_j)X_j \quad \text{for every } Y \in Der\,A. \tag{1}$$

The $A$-module $\Omega^q(A)$ consisting of (alternating) differential $q$-forms is also finitely generated projective. We will write $\Omega(A)$ for the finite direct sum $\sum_{q\geq 0} \Omega^q(A)$.

As discussed in [FL], we may define a regular affine domain $A$ to be symplectic when it is a Poisson algebra with $Der\,A = A \cdot Ham\,A$. In other words, $Der\,A$ is generated by $\{ham\,a |\, a \in A\}$ as an $A$-module. For any Poisson algebra there is an $A$-module map $\sharp : \Omega^1(A) \to Der\,A$ given by $(\sum a_i\,db_i)^\sharp = \sum a_i\,ham\,b_i$

([H]). To say that $A$ is symplectic is to assert that $\sharp$ has an inverse, which we denote $\flat$; indeed, a surjective map between finitely generated projective modules of the same rank must be an isomorphism. Now define $\omega : Der\,A \times Der\,A \to A$ by $\omega(X,Y) = \langle X, Y^\flat \rangle$. Then $\omega$ is an $A$-bilinear form. It is nondegenerate, for if $X \in Der\,A$ and $\omega(X,Y) = 0$ for all derivations $Y$ then $\langle X, da \rangle = 0$ for all $a \in A$. But $X(a) = 0$ for all $a$ forces $X = 0$. Less obviously, the form is alternating. It suffices to check this when $X = ham\,a$ and $Y = ham\,b$.

$$\omega(ham\,a, ham\,b) = \langle ham\,a, db \rangle = \{a,b\} = -\{b,a\} = -\omega(ham\,b, ham\,a).$$

We can now obtain a second formula for derivations, given an elite basis as described above. If $Y \in Der\,A$ then

$$Y^\flat = \sum_j \langle X_j, Y^\flat \rangle da_j.$$

Hence

$$Y = \sum_j \omega(X_j, Y) ham\,a_j \quad \text{for all } Y \in Der\,A. \tag{2}$$

Let $h$ be an indeterminate, so that $hA$ represents a free module of rank one. Set $L = hA \oplus Der\,A$, another projective $A$-module. More crucially, $L$ is a Lie algebra over $A$ under the bracket

$$[(ha, X), (hb, Y)] = h\omega(X,Y)$$

for $a, b \in A$ and $X, Y \in Der\,A$. Equivalently, $h$ is central in $L$ and $[X,Y] = h\omega(X,Y)$. We warn the reader of a treacherous difficulty with this notation. The commutator bracket on $Der\,A$ which arises from composition is entirely different. For the few times it will appear, we will use the notation $[X_\circ Y]$ for the derivation $X \circ Y - Y \circ X$. Returning to $L$, we can regard it as a graded Lie algebra by assigning nonzero members of $Der\,A$ the degree 1 and to $h$ the degree 2. Form the universal enveloping algebra $U_A(L)$. Since $L$ is a projective $A$-module, $U_A(L)$ has a Poincaré-Birkhoff-Witt theorem in the sense that the associated graded algebra for the filtration by products in $L$ is the symmetric $A$-algebra $S_A(L)$ ([R]). As in the case of enveloping algebras over fields ([D], 2.4), there is a homogeneous symmetrization map which injects $S_A(L)$ into $U_A(L)$ by sending $y_1 y_2 \cdots y_m$ in $S_A(L)$ with $y_j \in L$ to

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(m)}$$

in $U_A(L)$.

We are really interested in the subalgebra $S_A(Der\,A)$ of $S_A(L)$. This is due to the fact that factoring out $h$ yields a surjective $A$-algebra homomorphism $\pi : U_A(L) \to S_A(Der\,A)$. (The assertion is a consequence of the universal

properties of both $U_A(L)$ and $S_A(Der\,A)$.) Moreover, if $\pi^\dagger : S_A(Der\,A) \to U_A(L)$ denotes the restriction of the symmetrization map then $\pi^\dagger$ is an $A$-module homomorphism with $\pi \circ \pi^\dagger = id$. We denote by $S^\dagger_m$ the image in $U_A(L)$ of the m$^{\text{th}}$ homogeneous component of $S_A(Der\,A)$ under $\pi^\dagger$.

By applying induction on the length of words in $L$ to the formula $[X,Y] = h\omega(X,Y)$ for $X, Y \in Der\,A$ we find that $[x,y] \in hU_A(L)$ for all $x, y \in U_A(L)$. Consequently, $\frac{1}{h}[x,y]$ makes sense. This leads to a weak quantization result.

**Theorem 1.1** *The form $\omega$ extends uniquely to an $A$-bilinear Poisson bracket $\{*,*\}_\omega$ on $S_A(Der\,A)$ in such a way that for all $x, y \in U_A(L)$,*

$$\pi(\frac{1}{h}[x,y]) = \{\pi(x), \pi(y)\}_\omega.$$

**Proof:** It is easy to verify that the formula above can be used to define a binary operation on $S_A(L)$. It is also obvious that the resulting bracket is $A$-linear and satisfies the appropriate Liebniz and Lie requirements. Moreover, if $X, Y \in Der\,A$ then

$$\{X,Y\}_\omega = \{\pi(X), \pi(Y)\}_\omega = \pi(\frac{1}{h}[X,Y]) = \pi\omega(X,Y) = \omega(X,Y).$$

Since $Der\,A$ generates $S_A(Der\,A)$ as an $A$-algebra, the bracket's behavior on $Der\,A$ determines its behavior on the entire algebra. ■

Since $L$ is a graded Lie algebra, $U_A(L)$ is a graded associative $A$-algebra. This grading is compatible, under $\pi$ and $\pi^\dagger$, with the natural grading on $S_A(Der\,A)$. As a permanent notation we use $W$ for the completion of $U_A(L)$ and $S$ for the completion of $S_A(Der\,A)$ with respect to these gradings. With a slight abuse of notation we still have the completed maps $\pi$ and $\pi^\dagger$. (These functions commute with infinite sums. Without further notice, the term "homomorphism" will include this continuity property whenever it makes sense.) Let $W_p$ denote the p$^{\text{th}}$ homogeneous component of $W$ and write $W = \overline{\sum}_p W_p$ for the complete direct sum. Set $S^\dagger = \overline{\sum}_p S^\dagger_p$, an $A$-submodule of $W$. Observe that $S^\dagger_p \subseteq W_p$. More precisely, $\pi$ and $\pi^\dagger$ preserve degree.

The quantization of $A$ turns out to be the $k$-subalgebra of $W$ on which a certain derivation vanishes. This derivation, however, is defined on a slightly larger ring. Consider $W \otimes_A \Omega(A)$ and its shadow $S \otimes_A \Omega(A)$. (From now on we simply write $\Omega$ instead of $\Omega(A)$.) Both will be regarded as $A$-algebras in the sense of $\mathbf{Z}_2$-graded algebras. Thus if $x \in W \otimes_A \Omega^{q(1)}$ and $y \in W \otimes_A \Omega^{q(2)}$ then the graded commutator is defined to be

$$[x,y] = xy - (-1)^{q(1)q(2)} yx.$$

The center of $W \otimes \Omega$ consists of those elements $z$ such that $[z,t] = 0$ for all $t \in W \otimes \Omega$. This definition is designed so that $\Omega$ lies in the center of $W \otimes \Omega$.

Another easy consequence of the definition of graded bracket is that

$$[x, y] \in h(W \otimes \Omega)$$

for all $x, y \in W \otimes \Omega$. Finally, recall that a graded derivation $D$ from $W \otimes \Omega$ to itself of weight $N$ is a $k$-linear (continuous) map such that

$$D(W \otimes \Omega^q) \subseteq W \otimes \Omega^{q+N}$$

for all $q$, and

$$D(xy) = D(x)y + (-1)^q x D(y)$$

for $x \in W \otimes \Omega^q$ and arbitrary $y$.

Notice that $W \otimes \Omega$ is bigraded with homogeneous components $W_p \otimes \Omega^q$. We say that an element in this submodule has $W$-degree $p$ and $\Omega$-degree $q$. Since $z$ lies in the center of $W \otimes \Omega$ if and only if $[X, z] = 0$ for all $X \in Der\, A$, we see that each $W_p \otimes \Omega^q$ component of a central element is central.

Extend $\pi$ (resp. $\pi^\dagger$) to a map from $W \otimes \Omega$ to $S \otimes \Omega$ (resp. $S \otimes \Omega$ to $W \otimes \Omega$) by identifying it with $\pi \otimes id$ (resp. $\pi^\dagger \otimes id$).

**Theorem 1.2** *Every element of $W \otimes \Omega$ can be written uniquely in the form $\overline{\sum}_{m \geq 0} h^m s_m$ with $s_m \in S^\dagger \otimes \Omega$.*

**Proof** We first treat existence. It suffices to show by induction on $p$ that every element $u$ of $W_p$ has the form $\sum_{0 \leq m < p} h^m s_m$ with $s_m \in S^\dagger_{p-2m}$. If $p = 0$ then $W_0 = A$, so $u \in S^\dagger_0$; if $p = 1$ then $W_1 = Der\, A$, in which case $u \in S^\dagger_1$. In general,

$$u - \pi^\dagger \pi(u) \in W_p.$$

Since $\pi(u - \pi^\dagger \pi(u)) = 0$, we may write

$$u - \pi^\dagger \pi(u) = hv$$

for some $v \in W_{p-2}$. Apply induction to $v$.

As to uniqueness, we must show that if $\overline{\sum}_{m \geq 0} h^m s_m = 0$ with $s_m \in S^\dagger \otimes \Omega$ then $s_m = 0$ for all $m$. Since $U_A(L)$ is an integral domain (e.g., from the fact that $A$ is a domain and the PBW theorem), $W \otimes \Omega$ has no $h$-torsion. Thus if the sum is not formally zero, we may assume that $s_0$ is not zero.

$$0 = \pi(0) = \pi\left(\overline{\sum} h^m s_m\right) = \pi(s_0).$$

But $\pi$ is injective on $S^\dagger$. Hence $s_0 = 0$, a contradiction. ■

**Theorem 1.3** *Assume that $\Phi : W \otimes \Omega \to W \otimes \Omega$ and $\Phi_S : S \otimes \Omega \to S \otimes \Omega$ are graded derivations which satisfy*

$$\pi \circ \Phi = \Phi_S \circ \pi.$$

*If $\Phi(Der\, A) \subseteq \Omega$ and $\Phi(\Omega) \subseteq \Omega$ then*

$$\Phi \circ \pi^\dagger = \pi^\dagger \circ \Phi_S.$$

**Proof:** We will repeatedly use the observation that the commutative square for $\pi$ and $\Phi$ implies that $\Phi$ and $\Phi_S$ are formally the same on both $Der\, A$ and $\Omega$. It suffices to prove the desired equality when restricted to $S_p$ for all $p$. Indeed, if $c \in S_p$ and $\mu \in \Omega$ then

$$\begin{aligned}
\left(\Phi \circ \pi^\dagger\right)(c \otimes \mu) &= \Phi\left(\pi^\dagger(c)\mu\right) \\
&= \left(\Phi \circ \pi^\dagger\right)(c)\mu + (-1)^p \pi^\dagger(c)\Phi(\mu) \\
&= \left(\pi^\dagger \circ \Phi_S\right)(c)\mu + (-1)^p \pi^\dagger\left(c\, \Phi_S(\mu)\right) \\
&= \left(\pi^\dagger \circ \Phi_S\right)(c \otimes \mu).
\end{aligned}$$

So now consider $Y_1, \ldots, Y_m \in Der\, A$ and fix $r \in \{1, \ldots, m\}$. For each choice $\sigma \in \mathcal{S}_m$ exactly one of the $m$ terms in $\Phi(Y_{\sigma(1)} \cdots Y_{\sigma(m)})$ has a $\Phi(Y_r)$ appearing. Thus if we expand $\Phi\left(\sum_{\sigma \in \mathcal{S}_m} Y_{\sigma(1)} \cdots Y_{\sigma(m)}\right)$ we will get $m!$ terms with $\Phi(Y_r)$ as a factor. Now $\Phi(Y_r)$ commutes with all derivations of $A$ so we can write any such term as $P\Phi(Y_r)$ with $P$ a product of the $Y_i$ 's with $Y_r$ omitted. For each order of the factors in $P$ we get $m$ repetitions of this term (i.e., one for each original placement of $\Phi(Y_r)$ ). In other words, the contribution to

$$\Phi\left(\frac{1}{m!}\sum_\sigma Y_{\sigma(1)} \cdots Y_{\sigma(m)}\right)$$

of terms with the form $P\Phi(Y_r)$ is the symmetrization of $Y_1 \cdots \hat{Y}_r \cdots Y_m$ multiplied by $\Phi(Y_r)$. The restricted formula follows. ■

**Theorem 1.4** *The center of $W \otimes \Omega$ is $A[[h]] \otimes_A \Omega$.*

**Proof:** We already know that $A[[h]] \otimes_A \Omega$ is contained in the center. So suppose $\xi$ is a central element. Write

$$\xi = \overline{\sum_{m \geq 0}} h^m \xi_m$$

with $\xi_m \in S^\dagger \otimes \Omega$. We first argue that each $\xi_m$ lies in the center. This will follow if we show that $[Y, \xi_m] = 0$ for all $Y \in Der\, A$. The strategy is to use

Theorem 1.1 : $\frac{1}{h} ad\, Y$ is a graded derivation on $W \otimes \Omega$ and $ham_\omega Y$ is a graded derivation on $S \otimes \Omega$ such that

$$\pi \circ \frac{1}{h} ad\, Y = ham_\omega Y \circ \pi.$$

Moreover, $\left(\frac{1}{h} ad\, Y\right)(Der\, A) \subseteq \Omega^0$ and $\left(\frac{1}{h} ad\, Y\right)(\Omega) = 0$. Now apply Theorem 1.3.

$$\left(\frac{1}{h} ad\, Y\right) \circ \pi^\dagger = \pi^\dagger \circ ham_\omega Y.$$

In particular,

$$\left(\frac{1}{h} ad\, Y\right) \left(S^\dagger \otimes \Omega\right) \subseteq S^\dagger \otimes \Omega.$$

Thus

$$\overline{\sum_{m \geq 0}} h^m \left(\frac{1}{h} ad\, Y\right)(\xi_m)$$

is the unique $h$-series expansion for $\left(\frac{1}{h} ad\, Y\right)(\xi)$. On the other hand, $\left(\frac{1}{h} ad\, Y\right)(\xi) = 0$. Therefore each $\left(\frac{1}{h} ad\, Y\right)(\xi_m) = 0$. It follows that $\xi_m$ is central.

We have reduced the theorem to showing that if $\xi$ is central and lies in $S_p^\dagger \otimes \Omega^q$ then $\xi \in \Omega$. By applying induction to formula (2) (with respect to an elite basis $(X_1, a_1), \ldots, (X_n, a_n)$ for $Der\, A$), we may prove that if $Y_1, \ldots, Y_p \in Der\, A$ then

$$\sum_{j=1}^{n} [Y_1 Y_2 \cdots Y_p, X_j] ham\, a_j \equiv p Y_1 Y_2 \cdots Y_p \quad (mod\ h).$$

Thus if $u \in S_p^\dagger$ then

$$\sum_{j=1}^{n} [u, X_j] ham\, a_j \equiv pu \quad (mod\ h).$$

(Note that this vacuously covers the case $p = 0$.) Direct calculation shows that if $\nu \in \Omega^q$ we can extend the congruence,

$$\sum_{j=1}^{n} [u \otimes \nu, X_j] ham\, a_j \equiv p(u \otimes \nu) \quad (mod\ h).$$

If we apply this to $\xi$ we obtain

$$0 \equiv p\xi \quad (mod\ h).$$

In other words, either $p = 0$ or $\pi(\xi) = 0$. Of course, $\pi$ is injective on $S^\dagger \otimes \Omega$. We conclude that $p = 0$. ■

## 2 The differentials

The foundation of Fedosov's calculations is the existence of two differentials on $W \otimes \Omega$, extending $\sharp$ and $\flat$, which are "homotopy inverses" of each other. It will take us some technical preparation to establish an algebraic version.

Suppose that $\alpha : A \to \Omega$ is a derivation into an $A$-bimodule and that $f : Der\, A \to W \otimes \Omega$ is a $k$-linear transformation such that

$$f(aX) = \alpha(a)X + af(X), \quad \text{and}$$

$$h\alpha\left(\omega(X, Y)\right) = [f(X), Y] + [X, f(Y)]$$

for all $X, Y \in Der\, A$ and $a \in A$. Then $f$ and $\alpha$ extend to a single $k[h]$-linear derivation from $U_A(L)$ to $W \otimes \Omega$. The proof is a straightforward modification of the upper triangular trick which can be found in [J], page 154.

We first apply this to the situation in which $\alpha = 0$ and $f = \flat$. If $X \in Der\, A$ then $X^\flat \in \Omega^1$. Thus $X^\flat$ commutes with all elements of $W$ (and, in particular, with any $Y \in Der\, A$). The equations above are satisfied trivially. Hence $\flat$ extends to a derivation

$$\delta : U_A(L) \to W \otimes \Omega.$$

Since $\delta(W_p) \subseteq W_{p-1} \otimes \Omega^1$ it is easy to extend $\delta$ to a derivation

$$\delta : W \to W \otimes \Omega.$$

Finally, it extends to a graded derivation of weight 1 on all of $W \otimes \Omega$ by sending $u \otimes \nu$ to $\delta(u)\nu$. (The calculation rests on the observation that $\delta(W) \subseteq W \otimes \Omega^1$, so that if $\nu \in \Omega^q$ then $\nu\delta(u') = (-1)^q \delta(u')\nu$.) We call this derivation $\delta$ as well. Its crucial properties are that

$$\begin{aligned} &\delta(\Omega) = 0 \quad ; \\ &\delta(h) = 0 \quad ; \quad \text{and} \\ &\delta(X) = X^\flat \quad \text{for all } X \in Der\, A. \end{aligned}$$

With respect to the grading, $\delta(W_p \otimes \Omega^q) \subseteq W_{p-1} \otimes \Omega^{q+1}$.

In addition, $\delta^2 = 0$. Indeed, it certainly suffices to check that $\delta^2(W_p) = 0$ for all $p$. Now if $u, v \in W$ then

$$\begin{aligned} \delta^2(uv) &= \delta\left(\delta(u)v + u\delta(v)\right) \\ &= \left(\delta^2(u)v - \delta(u)\delta(v)\right) + \left(\delta(u)\delta(v) + u\delta^2(v)\right) \end{aligned}$$

because $\delta(u) \in W \otimes \Omega^1$ and $\delta$ is a graded derivation. We are reduced to checking that $\delta^2(Der\, A) = 0$. But this is immediate.

There is an induced derivation $\delta_S : S \otimes \Omega \to S \otimes \Omega$ which is $A$-linear and sends $X$ in $Der\, A$ to $X^\flat$. It is built so that $\pi \circ \delta = \delta_S \circ \pi$. We also have $\delta \circ \pi^\dagger = \pi^\dagger \circ \delta_S$ by virtue of Theorem 1.3.

The second differential will be based on a function which is defined on $\Omega$ and vanishes on $W$. We will need the classical contraction map on $\Omega(A)$. If $X \in Der\,A$ then $\iota_X : \Omega^q(A) \to \Omega^{q-1}(A)$ is an $A$-linear map given by

$$(\iota_X \xi)(Y_1, \ldots, Y_{q-1}) = \xi(X, Y_1, \ldots, Y_{q-1}).$$

The convention is that $\Omega^{-1} = 0$ and

$$\iota_X(\nu) = \langle X, \nu \rangle \quad \text{for } \nu \in \Omega^1.$$

Explicitly,

$$\iota_X(a_0 da_1 da_2 \cdots da_q) = \sum_{t=1}^{q} (-1)^{t-1} a_0 da_1 da_2 \cdots d\breve{a}_t \cdots da_q$$

where $d\breve{a}_t$ indicates that this factor is replaced with $X(a_t)$. It is standard that $\iota_X$ is a graded derivation from $\Omega$ to itself of weight -1.

**Lemma 2.1** *There is a unique $A$-linear map*

$$\delta^* : \Omega(A) \to Der\,A \otimes_A \Omega(A)$$

*such that $\delta^*(\nu) = \nu^\sharp$ for $\nu \in \Omega^1$ and*

$$\delta^*(\alpha\beta) = \delta^*(\alpha)\beta + (-1)^q \alpha \delta^*(\beta)$$

*for $\alpha \in \Omega^q$ and arbitrary $\beta \in \Omega$.*

**Proof:** Recall that we have a pairing on $Der\,A \otimes \Omega^1$ obtained from the $A$-module isomorphism

$$Der\,A \simeq Hom_A(\Omega^1, A).$$

Since $\Omega^q$ is a projective $A$-module, we may tensor and obtain

$$Der\,A \otimes_A \Omega^q \simeq Hom_A(\Omega^1, \Omega^q).$$

This is isomorphism can be described by an extension of the original pairing: if $X \in Der\,A,\ \nu \in \Omega^q$, and $\lambda \in \Omega^1$ define

$$\langle X \otimes \nu, \lambda \rangle' = \langle X, \lambda \rangle \nu.$$

Now if $\phi \in Hom_A(\Omega^1, \Omega^q)$ there exists a unique $f \in Der\,A \otimes \Omega^q$ such that $\phi(\lambda) = \langle f, \lambda \rangle'$ for all $\lambda \in \Omega^1$. The new pairing has an additional attractive property. If $\mu \in \Omega^r$ and $g \in Der\,A \otimes \Omega^q$ then

$$\mu \langle g, \lambda \rangle' = \langle \mu g, \lambda \rangle' \quad \text{and} \quad \langle g, \lambda \rangle' \mu = \langle g\mu, \lambda \rangle'.$$

Fix $\nu \in \Omega^{q+1}$. Consider the function which sends $\lambda$ in $\Omega^1$ to $-\iota_{\lambda^\sharp}(\nu)$; it lies in $Hom_A(\Omega^1, \Omega^q)$. Thus there exists an element we will call $\delta^*(\nu)$ in $Der\,A \otimes \Omega^q$ such that

$$\langle \delta^*(\nu), \lambda \rangle' = -\iota_{\lambda^\sharp}\nu \quad \text{for all } \lambda \in \Omega^1.$$

Clearly $\delta^*$ is $A$-linear; we posit $\delta^*(A) = 0$. If $\nu \in \Omega^1$ then

$$\begin{aligned} \langle \delta^*(\nu), \lambda \rangle &= -\iota_{\lambda^\sharp}\nu \\ &= -\langle \lambda^\sharp, \nu \rangle \\ &= -\omega(\lambda^\sharp, \nu^\sharp) \\ &= \omega(\nu^\sharp, \lambda^\sharp) \\ &= \langle \nu^\sharp, \lambda \rangle. \end{aligned}$$

Hence $\delta^*$ agrees with $\sharp$ on $\Omega^1$.

Finally, we shall prove that $\delta^*$ has the derivation-like property because $\iota_{\lambda^\sharp}$ is a graded derivation. We borrow the symbols $\alpha$ and $\beta$ from the statement of the lemma and assume $\beta$ is homogeneous.

$$\begin{aligned} \langle \delta^*(\alpha\beta), \lambda \rangle' &= \iota_{\lambda^\sharp}(\alpha\beta) \\ &= \iota_{\lambda^\sharp}(\alpha)\beta + (-1)^q \alpha \iota_{\lambda^\sharp}(\beta) \\ &= \langle \delta^*(\alpha), \lambda \rangle' \beta + (-1)^q \alpha \langle \delta^*(\beta), \lambda \rangle' \\ &= \langle \delta^*(\alpha)\beta + (-1)^q \alpha \delta^*(\beta), \lambda \rangle'. \quad \blacksquare \end{aligned}$$

**Lemma 2.2** $(\delta_S^*)^2 = 0$.

**Proof:** It suffices to prove that $(\delta_S^*)^2(\Omega^q) = 0$ for all $q$. We proceed by induction on $q$. The equality is true for $q = 0$ and $q = 1$ by degree considerations. The induction step amounts to showing that if $\nu \in \Omega^q$, $(\delta_S^*)^2(\nu) = 0$ and $b \in A$ then $(\delta_S^*)^2(\nu\, db) = 0$.

$$\begin{aligned} (\delta_S^*)^2(\nu\, db) &= \delta_S^*(\delta_S^*(\nu) db + (-1)^q \nu \delta_S^*(db)) \\ &= (\delta_S^*)^2(\nu) db + (-1)^{q-1} \delta_S^*(\nu)\delta_S^*(db) + (-1)^q \delta_S^*(\nu)\delta_S^*(db) + \\ &\qquad\qquad (-1)^{2q}(\delta_S^*)^2(db) \\ &= 0. \quad \blacksquare \end{aligned}$$

**Lemma 2.3** *If $x \in S_p \otimes \Omega^q$ then*

$$(\delta_S \delta_S^* + \delta_S^* \delta_S)(x) = (p+q)x.$$

**Proof:** We first check this for $x \in S_p$. Since $\delta_S^*$ vanishes on $S$, we require $\delta_S^* \delta_S(Y_1 \cdots Y_p) = pY_1 \cdots Y_p$ for $Y_j \in Der\, A$. Calculate:

$$\begin{aligned}\delta_S^* \delta_S(Y_1 \cdots Y_p) &= \delta_S^* \left( \sum_{j=1}^{p} Y_1 \cdots \hat{Y}_j \cdots Y_p Y_j^{\flat} \right) \\ &= \sum_{j=1}^{p} Y_1 \cdots \hat{Y}_j \cdots Y_p Y_j^{\flat\sharp} \\ &= pY_1 \cdots Y_p.\end{aligned}$$

(Though the role of commutativity is crucial here, we will see in the next lemma how to lift the calculation to $W$.)

Next we plug in $\mu \in S^q$. This time we recall that $\delta_S(\mu) = 0$. The formula $\delta_S \delta_S^*(\mu) = qu$ is proved inductively on $q$. For $q = 1$,

$$\delta_S \delta_S^*(\mu) = \delta_S(\mu^{\sharp}) = \mu^{\sharp\flat} = \mu.$$

If $\delta_S \delta_S^*(\mu) = qu$ and $b \in A$ then

$$\begin{aligned}\delta_S \delta_S^*(\mu\, db) &= \delta_S \left( \delta_S^*(\mu) db + (-1)^q \mu (db)^{\sharp} \right) \\ &= \delta_S \delta_S^*(\mu) db + (-1)^{2q} \mu (db)^{\sharp\flat} \\ &= (q+1)\mu\, db.\end{aligned}$$

Last of all, we evaluate $\delta_S \delta_S^* + \delta_S^* \delta_S$ on $y \otimes \nu$ for $y \in S_p$ and $\nu \in \Omega^q$.

$$\begin{aligned}&\left(\delta_S \delta_S^* + \delta_S^* \delta_S\right)(y \otimes \nu) \\ &= \delta_S\left(y \delta_S^*(\nu)\right) + \delta_S^*\left(\delta_S(y)\nu\right) \\ &= \delta_S(y)\delta_S^*(\nu) + y\delta_S\delta_S^*(\nu) + \delta_S^*\delta_S(y)\nu + (-1)\delta_S(y)\delta_S^*(\nu) \\ &= qy\nu + py\nu. \;\blacksquare\end{aligned}$$

We shall jack Lemma 2.3 up to $W \otimes \Omega$. It turns out that the formula in the lemma characterizes $S^{\dagger}$.

**Proposition 2.1** *Let $x \in W_p \otimes \Omega^q$. Then $x \in S_p^{\dagger} \otimes \Omega^q$ if and only if*

$$(\delta^* \delta + \delta \delta^*)(x) = (p+q)x.$$

**Proof:** To prove the formula for $x \in S_p^{\dagger} \otimes \Omega^q$ it suffices to verify it when $x \in S_p^{\dagger}$ — the rest of the proof of Lemma 2.3 holds for $\delta^*$ and $\delta$ replacing $\delta_S^*$ and $\delta_S$ respectively. So we may assume that $x = \frac{1}{p!} \sum_{\sigma \in \mathcal{S}_p} Y_{\sigma(1)} \cdots Y_{\sigma(p)}$ with

$Y_1, \ldots, Y_p$ in $Der\, A$. For each $\sigma \in \mathcal{S}_p$

$$\begin{aligned}\delta^*\delta(Y_{\sigma(1)} \cdots Y_{\sigma(p)}) &= \delta^* \left( \sum_{j=1}^{p} Y_{\sigma(1)} \cdots \hat{Y}_{\sigma(j)} \cdots Y_{\sigma(p)} Y^{\flat}_{\sigma(j)} \right) \\ &= \sum_{j=1}^{p} Y_{\sigma(1)} \cdots \hat{Y}_{\sigma(j)} \cdots Y_{\sigma(p)} Y_{\sigma(j)}.\end{aligned}$$

Thus each original permutation of $Y_1, \ldots, Y_p$ gives rise to $p$ permutations, each ending with a different choice of $Y_t$. From the opposite point of view, if we take any product ending in $Y_t$, the last letter could have moved to the end position (via the $\delta^*$ shuffle) from any one of $p$ positions. Thus each permutation appears $p$ times as we list the terms in the expansion of $\delta^*\delta \left( \sum_{\sigma \in \mathcal{S}_p} Y_{\sigma(1)} \cdots Y_{\sigma(p)} \right)$. In other words, $\delta^*\delta(x) = px$.

Conversely, assume that $x \in W_p \otimes \Omega^q$ satisfies the formula. Write $x = \overline{\sum}_{m \geq 0} h^m s_{p-2m}$ where $s_t \in S_t^{\dagger} \otimes \Omega^q$. Then

$$(\delta^*\delta + \delta\delta^*)(x) = \overline{\sum_{m \geq 0}} h^m (p + q - 2m) s_{p-2m}$$

according to the previous direction of the proposition. By the uniqueness of the $h$-series,

$$p + q = p + q - 2m$$

for all $m$ with $s_{p-2m} \neq 0$. That is, $m = 0$. ■

We conclude this section with Fedosov's fundamental "homotopy inverse" relation. Unfortunately, this will involve one further layer of notation because $\delta^*$ does not quite meet all of his requirements. We perturb $\delta^*$ in defining a new function $\tilde{\delta} : W \otimes \Omega \to W \otimes \Omega$.

If $a \in A$ set $\tilde{\delta}(a) = 0$. If $x \in S_p^{\dagger} \otimes \Omega^q$ for $(p, q) \neq 0$ set

$$\tilde{\delta}(x) = \frac{1}{p+q} \pi^{\dagger} \delta_S^* \pi.$$

Extend $\tilde{\delta}$ $A[[h]]$-linearly to all of $W \otimes \Omega$ using the $h$-series. Note that $\tilde{\delta}(W_p \otimes \Omega^q) \subseteq W_{p+1} \otimes \Omega^{q-1}$ and $\tilde{\delta}(W) = 0$. We still have $\tilde{\delta}(\nu) = \nu^{\sharp}$ for $\nu \in \Omega^1$.

Consider the augmentation map from $S$ to $A$. It has two relevant extensions to all of $S \otimes \Omega$. The first, denoted $\tau_S : S \otimes \Omega \to A$, is the composition of the projection $S \otimes \Omega \to S$ with the augmentation map. The second, denoted $T_S : S \otimes \Omega \to \Omega$, is the augmentation map tensored with the identity. Thus $T_S = (\tau_S | S) \otimes id$. Both $T_S$ and $\tau_S$ are $A$-linear maps which are, in a sense, idempotent.

If $x \in W \otimes \Omega$, decompose $x = \overline{\sum}_{m \geq 0} h^m s_m$ with $s_m \in S^\dagger \otimes \Omega$ and define $\tau : W \otimes \Omega \to A[[h]]$ by

$$\tau(x) = \overline{\sum_{m \geq 0}} h^m \tau_S \pi(s_m).$$

Here $\tau_S \pi$ can also be regarded as the composition of the projection $W \otimes \Omega \to W$ with the augmentation map $W \to A$. The uniqueness of the $h$-series implies that $\tau$ is $A[[h]]$-linear. Set $T : W \otimes \Omega \to A[[h]] \otimes_A \Omega$ to be $(\tau|W) \otimes id$. We remark that $\tau$ is somewhat mysterious. For example, if $X$ and $Y$ are derivations of $A$ then $XY = \frac{1}{2}(XY + YX) + \frac{1}{2}[X, Y]$. Hence

$$\tau(XY) = \frac{1}{2} h \omega(X, Y).$$

It immediately follows from the definitions that

$$\pi \circ \tau = \tau_S \circ \pi \quad \text{and} \quad \pi \circ T = T_S \circ \pi.$$

**Theorem 2.1** *(i)* $\tilde{\delta}^2 = 0$.

*(ii) (Fundamental Formula)* $\tilde{\delta}\delta + \delta\tilde{\delta} + \tau = id$.

**Proof:** Since all of the maps in the statement of the theorem are $A[[h]]$-linear, it suffices to verify the equalities when restricted to $S_p^\dagger \otimes \Omega^q$. Let $x$ be in this component.

$$\begin{aligned} \tilde{\delta}\left(\tilde{\delta}(x)\right) &= \tilde{\delta}\left(\frac{1}{p+q} \pi^\dagger \delta_S^* \pi(x)\right) \\ &= \left(\frac{1}{p+q}\right)^2 \pi^\dagger \delta_S^* \pi \pi^\dagger \delta_S^* \pi(x). \end{aligned}$$

Since $\pi\pi^\dagger = id$ and $(\delta_S^*)^2 = 0$ (by Lemma 2.2), the last expression is zero. Thus $\tilde{\delta}^2(x) = 0$.

As to $(ii)$, if $(p, q) = 0$ then $x \in A$. In this case, $\delta(x) = \tilde{\delta}(x) = 0$. Since $\tau|A = id$,

$$\tilde{\delta}\delta(x) + \delta\tilde{\delta}(x) + \tau(x) = x.$$

If $(p, q) \neq 0$ then $\tau(x) = 0$.

$$\begin{aligned} (\tilde{\delta}\delta + \delta\tilde{\delta})(x) &= \frac{1}{p+q}\left(\pi^\dagger \delta_S^* \pi \delta + \delta \pi^\dagger \delta_S^* \pi\right)(x) \\ &= \frac{1}{p+q}\left(\pi^\dagger \delta_S^* \delta_S \pi + \pi^\dagger \delta_S \delta_S^* \pi\right)(x) \end{aligned}$$

by the commuting squares for $\delta$ with $\pi$ and $\pi^\dagger$. Invoking Lemma 2.3 we obtain

$$(\tilde{\delta}\delta + \delta\tilde{\delta})(x) = \pi^\dagger \pi(x).$$

Since $x \in S^\dagger \otimes \Omega$ we have $\pi^\dagger \pi(x) = x$. ■

## 3 Connections

A connection for $A$ associates to each pair of derivations $X, Y \in \mathit{Der}\, A$ a third derivation $\nabla_X Y$ with the properties that $\nabla$ is $A$-linear in the lower argument and

$$\nabla_X \, aY = X(a)Y + a\nabla_X Y \tag{3}$$

for all $a \in A$. We will initially think of $\nabla Y$ as a function from $\mathit{Der}\, A$ to itself via $(\nabla Y)(X) = \nabla_X Y$. Given an elite basis $(X_1, a_1), \ldots, (X_n, a_n)$ for $\mathit{Der}\, A$ then

$$\nabla_X Y = \nabla_{\sum X(a_i) X_i} Y = \sum X(a_i) \nabla_{X_i} Y.$$

In this way we can identify $\nabla Y$ with $\sum \nabla_{X_i} Y \otimes da_i$ in $\mathit{Der}\, A \otimes \Omega^1$ and regard $\nabla$ as a function $\mathit{Der}\, A \to \mathit{Der}\, A \otimes \Omega^1$.

It is tempting to try to extend $\nabla$ to a $k$-endomorphism of $W \otimes \Omega$. The classical insight is to define $\nabla a$ to be $da$ for $a \in A$. Then (3) becomes

$$\nabla a Y = (\nabla a) Y + a \nabla Y.$$

We are now on familiar ground. The discussion at the beginning of the second section tells us that $\nabla$ extends to a $k[[h]]$-linear derivation from $W$ into $W \otimes \Omega^1$ provided

$$h \, d\omega(X, Y) = [\nabla X, Y] + [X, \nabla Y]$$

in $W \otimes \Omega^1$.

Recall that a connection $\nabla$ for $A$ is *parallel* (to the symplectic form $\omega$) when

$$Z\left(\omega(X, Y)\right) = \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y)$$

for all $X, Y, Z \in \mathit{Der}\, A$. It is a *Poisson connection* if it also satisfies the "no torsion" condition

$$\nabla_X Y - \nabla_Y X = [X_\circ Y].$$

An extensive algebraic discussion of Poisson connections can be found in [F2]. In particular, a regular symplectic affine domain such as $A$ always supports a Poisson connection.

**Lemma 3.1** *If $\nabla$ is a parallel connection then*

$$h \, d\omega(X, Y) = [\nabla X, Y] + [X, \nabla Y]$$

*for all $X, Y \in \mathit{Der}\, A$.*

**Proof:** We expand $[\nabla X, Y] + [X, \nabla Y]$ using the formula

$$\nabla V = \sum (\nabla_{X_i} V) \, da_i$$

for $V = X$ and $V = Y$.

$$
\begin{aligned}
[\nabla X, Y] + [X, \nabla Y] &= \sum_i h\left(\omega(\nabla_{X_i} X, Y) + \omega(X, \nabla_{X_i} Y)\right) da_i \\
&= \sum_i h X_i(\omega(X, Y)) da_i \\
&= h \sum_i \langle X_i, d\omega(X, Y)\rangle da_i \\
&= h\, d(\omega(X, Y)). \quad \blacksquare
\end{aligned}
$$

Assume from now on that $\nabla$ is a Poisson connection. The induced derivation $\nabla : W \to W \otimes \Omega^1$ extends to a graded derivation $\nabla : W\otimes \to W \otimes \Omega$ of weight 1 by setting

$$\nabla(x \otimes \nu) = (\nabla x)\nu + x\, d\nu.$$

(That the extension is well-defined follows from the centrality of $a$ and $da$ for $a \in A$.) The Poisson connection has a similar extension ${}_S\nabla : S \otimes \Omega \to S \otimes \Omega$ which is related to $\nabla$ by the commutative square ${}_S\nabla \circ \pi = \pi \circ \nabla$. We will need several less trivial identities.

**Lemma 3.2** *If $\nabla$ is a Poisson connection for $A$ then*

$$\omega(X, \nabla_Z ham\, u) = \omega(Z, \nabla_X ham\, u)$$

*for all $X, Z \in Der\, A$ and $u \in A$.*

**Proof:**

$$\omega(X, \nabla_Z ham\, u) = Z(X(u)) - \omega(\nabla_Z X, ham\, u)$$

because $\nabla$ is parallel. Since $\nabla$ has no torsion,

$$
\begin{aligned}
\omega(\nabla_Z X, ham\, u) &= \omega(\nabla_X Z, ham\, u) + \omega([Z_\circ X], ham\, u) \\
&= \omega(\nabla_X Z, ham\, u) + Z(X(u)) - X(Z(u)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\omega(X, \nabla_Z ham\, u) &= X(Z(u)) - \omega(\nabla_X Z, ham\, u) \\
&= \omega(Z, \nabla_X ham\, u)
\end{aligned}
$$

by one more application of parallelism. $\blacksquare$

**Lemma 3.3** *$\delta\nabla$ vanishes on $Ham\, A$.*

**Proof:** We re-examine the previous lemma with the help of a projective basis $(da_1, X_1), \ldots, (da_n, X_n)$ for $\Omega^1$. First of all, if $Y, Z \in Der\, A$ and $u \in A$ then

$$
\begin{aligned}
\langle Y, \delta(\nabla_Z ham\, u)\rangle &= \langle Y, (\nabla_Z ham\, u)^\flat\rangle \\
&= \omega(Y, \nabla_Z ham\, u) \\
&= \omega(Z, \nabla_Y ham\, u)
\end{aligned}
$$

with the last equality from Lemma 3.2. Now

$$\nabla_Y ham\, u = \sum_i Y(a_i)\nabla_{X_i} ham\, u.$$

Therefore

$$\langle Y, \delta(\nabla_Z ham\, u)\rangle = \langle Y, \sum_i \omega(Z, \nabla_{X_i} ham\, u) da_i\rangle.$$

Consequently

$$\delta(\nabla_Z ham\, u) = \sum_i \langle Z, \delta(\nabla_{X_i} ham\, u)\rangle da_i. \qquad (*)$$

Using the induced elite basis a second time, we recall that $\nabla ham\, u = \sum_j (\nabla_{X_j} ham\, u) da_j$, so

$$\begin{aligned}
\delta\nabla ham\, u &= \sum_j \delta(\nabla_{X_j} ham\, u) da_j \\
&= \sum_j \sum_i \langle X_j, \delta(\nabla_{X_i} ham\, u)\rangle da_i da_j \quad \text{by } (*) \\
&= -\sum_i \left( \sum_j \langle X_j, \delta(\nabla_{X_i} ham\, u)\rangle da_j \right) da_i \\
&= -\sum_i \delta(\nabla_{X_i} ham\, u) da_i \quad \text{by the basis property} \\
&= -\delta\nabla ham\, u. \ \blacksquare
\end{aligned}$$

**Theorem 3.1** $\delta\nabla + \nabla\delta = 0$

**Proof:** Notice that $(\delta\nabla + \nabla\delta)(\xi) = 0$ for $\xi \in \Omega$ because $\delta$ vanishes on $\Omega$. If $\xi = ham\, u$ for $u \in A$ then

$$(\nabla\delta)(ham\, u) = \nabla(du) = 0.$$

Thus Lemma 3.3 implies that $(\delta\nabla + \nabla\delta)(\xi) = 0$ for this choice of $\xi$ as well. The value of these observations is that $\Omega$ and $Ham\, A$ generate $W \otimes \Omega$ as a $k[[h]]$-algebra (in the sense that they algebraically generate its homogeneous components).

The theorem will be completed once we prove that if $r$ and $t$ are bihomogeneous elements of $W \otimes \Omega$ with

$$(\delta\nabla + \nabla\delta)(r) = (\delta\nabla + \nabla\delta)(t) = 0$$

then

$$(\delta\nabla + \nabla\delta)(rt) = 0.$$

But a direct calculation shows that

$$(\delta\nabla + \nabla\delta)(rt) = (\nabla\delta(r) + \delta\nabla(r))t + r(\nabla\delta(t) + \delta\nabla(t)). \quad \blacksquare$$

**Theorem 3.2** $T \circ \nabla = \nabla \circ T$

**Proof:** By modifying the argument in Theorem 1.3 or Proposition 2.1 the reader may verify that $\pi^\dagger \circ \nabla = {}_S\nabla \circ \pi^\dagger$. (The bookkeeping will look familiar by writing $\nabla Y_t = \sum_j \nabla_{X_j} Y_t \otimes da_j$ with respect to an elite basis.)

Since $T$ and $\nabla$ are both $k[[h]]$-linear, it suffices to prove that $T \circ \nabla \circ \pi^\dagger = \nabla \circ T \circ \pi^\dagger$. A quick glance at the definition shows that $T \circ \pi^\dagger = T_S$. We are reduced to checking that $T_S \circ {}_S\nabla = {}_S\nabla \circ T$. This is an immediate consequence of the facts that

$$\begin{gathered} {}_S\nabla(S_p \otimes \Omega^q) \subseteq S_p \otimes \Omega^{q+1} \quad \text{and} \\ T_S | S_p \otimes \Omega^q = \begin{cases} 0 & \text{if } p \neq 0 \\ id & \text{if } p = 0\ . \end{cases} \quad \blacksquare \end{gathered}$$

The last general property we will need about connections is really an observation about curvature. It turns out that we do not rely on any geometric properties of curvature other than that it exists in a very weak algebraic manner of speaking: $\nabla^2$ is essentially inner. This is a special case of a more general phenomenon, based on the simple calculation that $\nabla^2$ is a graded $k[[h]]$-linear derivation of $W \otimes \Omega$ which vanishes on $\Omega$.

**Theorem 3.3** *Let $D : W \otimes \Omega \to W \otimes \Omega$ be a graded derivation such that $D(W_1) \subseteq W_1 \otimes \Omega$. Then $D$ vanishes on $A[[h]] \otimes \Omega$ if and only if there exists a $\Gamma \in W \otimes \Omega$ such that $D = \frac{1}{h} ad\, \Gamma$ (i.e., $D(\alpha) = \frac{1}{h}[\Gamma, \alpha]$ for all $\alpha \in W \otimes \Omega$ ).*

**Proof:** Since $A[[h]] \otimes \Omega$ is the center of $W \otimes \Omega$, any inner derivation of $W \otimes \Omega$ vanishes on $A[[h]] \otimes \Omega$. Conversely, assume that $D$ vanishes on the center and set

$$\Gamma = -\frac{1}{2} \sum_i (ham\, a_i) DX_i$$

with respect to an elite basis. It suffices to show that $D(Y) = \frac{1}{h}[\Gamma, Y]$ for $Y \in Der\, A$. By definition,

$$[\Gamma, Y] = -\frac{1}{2} \sum_i [ham\, a_i, Y] DX_i - \frac{1}{2} \sum_i (ham\, a_i)[DX_i, Y].$$

We examine the first term.

$$\begin{aligned}
\sum_i [ham\, a_i, Y] DX_i &= h \sum_i \omega(ham\, a_i, Y) DX_i \\
&= -h \sum_i Y(a_i) DX_i \\
&= -hD\left(\sum_i Y(a_i) X_i\right) \quad \text{because } D(A) = 0, \\
&= -hDY.
\end{aligned}$$

Thus

$$[\Gamma, Y] = \frac{1}{2} hDY - \frac{1}{2} \sum_i (ham\, a_i)[DX_i, Y].$$

We now simplify the second term.

$$\begin{aligned}
\sum_i (ham\, a_i)[DX_i, Y] &= \sum_i (ham\, a_i) D\left([X_i, Y]\right) - \sum_i (ham\, a_i)[X_i, DY] \\
&= -\sum_i (ham\, a_i)[X_i, DY]
\end{aligned}$$

because $D(A) = 0$. According to the hypothesis of the theorem, $DY$ is a sum of expressions of the form $Z \otimes \xi$ with $Z \in Der\, A$ and $\xi \in \Omega$.

$$\begin{aligned}
\sum_i (ham\, a_i)[X_i, Z \otimes \xi] &= \sum_i (ham\, a_i)[X_i, Z] \otimes \xi \\
&= hZ \otimes \xi \quad \text{by (2)}.
\end{aligned}$$

Hence

$$\sum_i (ham\, a_i)[DX_i, Y] = -hDY.$$

We conclude that

$$[\Gamma, Y] = \frac{1}{2} hDY + \frac{1}{2} hDY = hDY. \quad \blacksquare$$

**Corollary 3.1** *Let $\nabla$ be a Poisson connection for $A$. Then there exists $R \in W_2 \otimes \Omega^2$ with $\nabla^2 = \frac{1}{h} ad\, R$.* $\blacksquare$

# 4 The Fedosov calculus

One of Fedosov's insights is to expand the notion of connection to any graded derivation on $W \otimes \Omega$ of weight one which formally satisfies equation (3). This

creates a source of connections large enough to always include one with no "curvature".

We begin with the technical engine which drives Fedosov's calculus.

**Definition 1** *A function* $\Phi : W \otimes \Omega \to W \otimes \Omega$ *is* **monotone** *provided*

$$\Phi\left(\overline{\sum_{p\geq t}} W_p \otimes \Omega\right) \subseteq \overline{\sum_{p\geq t}} W_p \otimes \Omega$$

*for all* $t \in \mathbf{N}$.

**Theorem 4.1 (Vanishing Theorem)** *Assume* $\Phi$ *is monotone. If* $\beta \in W \otimes \Omega$ *satisfies*

*(i)* $\delta(\beta) = \Phi(\beta)$*,*

*(ii)* $\tilde{\delta}(\beta) = 0$*, and*

*(iii)* $\tau(\beta) = 0$

*then* $\beta = 0$*.*

**Proof:** Assume $\beta \neq 0$. Write $\beta = \overline{\sum}_{p\geq t} b_p$ with $b_p \in W_p \otimes \Omega$ and $b_p \neq 0$. By the fundamental formula of Theorem 2.1,

$$\beta = \tilde{\delta}\delta(\beta) + \delta\tilde{\delta}(\beta) + \tau(\beta) = \tilde{\delta}\Phi(\beta).$$

Hence

$$\overline{\sum_{p\geq t}} b_p \in \overline{\sum_{p\geq t}} \tilde{\delta}(W_p \otimes \Omega).$$

But $\tilde{\delta}(W_p \otimes \Omega) \subseteq W_{p+1} \otimes \Omega$. That is,

$$\overline{\sum_{p\geq t}} b_p \in \overline{\sum_{p\geq t}} W_{p+1} \otimes \Omega.$$

We have reached the contradiction $b_t = 0$. ■

**Definition 2** *A* **Fedosov connection** $D$ *for* $A$ *is a graded* $k[[h]]$*-linear derivation of weight one on* $W \otimes \Omega$ *such that*

$$D(a) = da \text{ for all } a \in A;$$
$$D + \delta \text{ is monotone; and}$$
$$D^2 = 0.$$

**Theorem 4.2** *Assume $D$ is a Fedosov connection for $A$. Then for each $u \in A[[h]]$ there exists a unique $b \in W$ such that $D(b) = 0$ and $\tau(b) = u$.*

**Proof:** We first treat uniqueness. Suppose $D(b) = 0$ and $\tau(b) = 0$. Then $\delta(b) = (D+\delta)(b)$ and $\tilde{\delta}(b) = 0$ because $\tilde{\delta}$ is zero on $W$. The Vanishing Theorem yields $b = 0$.

As to existence, define a sequence $b_0, b_1, \ldots$ in $W$ by

$$b_0 = u \quad \text{and} \quad b_{m+1} = \tilde{\delta}(D+\delta)(b_m).$$

Since $\tilde{\delta}(D+\delta)\left(\overline{\sum_{p\geq t} W_p \otimes \Omega}\right) \subseteq \overline{\sum_{p\geq t+1} W_p \otimes \Omega}$, the series $\sum_{j=0}^{\infty} b_j$ converges. Call this sum $b$. By construction,

$$b = u + \tilde{\delta}(D+\delta)(b).$$

Observe that

$$\begin{aligned}
\tau(b) &= b - \tilde{\delta}\delta(b) \\
&= u + \tilde{\delta}(D+\delta)(b) - \tilde{\delta}\delta\tilde{\delta}(D+\delta)(b) \\
&= u + \tilde{\delta}(D+\delta)(b) - \tilde{\delta}(id - \tilde{\delta}\delta - \tau)(D+\delta)(b) \\
&= u + \tilde{\delta}(D+\delta)(b) - \tilde{\delta}(D+\delta)(b) + \tilde{\delta}\tau(D+\delta)(b).
\end{aligned}$$

But $(D+\delta)(b) \in W \otimes \Omega^1$ and $\tau$ is zero on $W \otimes \Omega^1$. Hence

$$\tau(b) = u.$$

We conclude that

$$b = \tau(b) + \tilde{\delta}(D+\delta)(b). \tag{$*$}$$

The argument is completed by showing that $D(b) = 0$ via $(*)$ and the Vanishing Theorem.

$$\delta D(b) = (D+\delta)D(b) - D^2(b) = (D+\delta)D(b).$$

This establishes condition $(i)$ of Theorem 4.1 for $D(b)$. By $(*)$, $b = \tau(b) + \tilde{\delta}\delta(b) + \tilde{\delta}D(b)$. With the fundamental formula, this yields

$$\delta\tilde{\delta}(b) = \tilde{\delta}D(b).$$

Since $\tilde{\delta}(W) = 0$, we obtain $(ii)$. Finally, $\tau D(b) = 0$ because $D(b) \in W \otimes \Omega^1$. ■

**Corollary 4.1** *Given a Fedosov connection $D$ for $A$, set*

$$B = \{b \in W | D(b) = 0\},$$

*a subalgebra of $W$ which contains $h$. Then the restriction of the augmentation map $\varepsilon : W \to A$ induces an isomorphism*

$$B/hB \simeq A$$

*and $\frac{1}{h}[b, b']$ is sent to $\{\varepsilon(b), \varepsilon(b')\}$. (In other words, $B$ is a naive quantization of $A$.)*

**Proof:** We first argue that $hB = B \cap \mathrm{Ker}\varepsilon$. Since $\varepsilon(h) = 0$, one inclusion is obvious. So suppose $b \in B \cap \mathrm{Ker}\varepsilon$. Using the $h$-series expansion,

$$b = \overline{\sum_{m\geq 0}} h^m s_m \quad \text{with} \quad s_m \in S^\dagger,$$

we see that $0 = \varepsilon(b) = \varepsilon(s_0)$. Hence

$$\tau(b) = \overline{\sum_{m\geq 1}} h^m \varepsilon(s_m) = h \overline{\sum_{m\geq 1}} h^{m-1}\varepsilon(s_m).$$

Since $\overline{\sum}_{m\geq 1} h^{m-1}\varepsilon(s_m) \in A[[h]]$, the theorem tells us that there exists $c \in B$ with $\tau(c) = \overline{\sum}_{m\geq 1} h^{m-1}\varepsilon(s_m)$. Therefore

$$hc \in B \quad \text{and} \quad \tau(hc) = \tau(b).$$

The uniqueness assertion of the theorem establishes $hc = b$. This proves $b \in hB$.

The restriction of $\varepsilon$ to $B$ is injective. It is surjective by the existence portion of the theorem: for each $u \in A$ there is an element $u + \overline{\sum}_{i>0} b_i$ in $B$ with $\varepsilon\left(\overline{\sum}_{i>0} b_i\right) = 0$ by virtue of the $W$-degrees involved. We now have

$$B/hB \simeq A.$$

For the Poisson bracket calculation, we may assume that $\tau(b) = \varepsilon(b)$ and $\tau(b') = \varepsilon(b')$. The $W_1$-component of $b$ is

$$\tilde{\delta}\nabla\varepsilon(b) = \tilde{\delta}(d\varepsilon(b)) = ham\,(\varepsilon(b)).$$

It follows that the $W_2$-component of $[b, b']$ is

$$h\omega(ham\,\varepsilon(b), ham\,\varepsilon(b')).$$

But $\omega(ham\,u, ham\,v) = \{u, v\}$ in general. Hence the $W_0$-component of $\frac{1}{h}[b, b']$ is $\{\varepsilon(b), \varepsilon(b')\}$. ■

# 5 Existence of Fedosov connections

The last step in [Fe] is the construction of a Fedosov connection. Here we follow the original quite closely, replacing classical identities about curvature with simple algebraic calculations. The overall strategy is transparent: given a Poisson connection, find a correction term $\gamma \in \overline{\sum}_{p\geq 3} W_p \otimes \Omega^1$ so that $\nabla - \delta + \frac{1}{h} ad\,\gamma$ is a Fedosov connection. Notice that this expression is a sum of graded derivations of weight 1, which makes it a derivation of the same type. Moreover,

$$(\nabla - \delta + \frac{1}{h} ad\,\gamma)(a) = da \quad \text{for all } a \in A$$

because $\delta$ vanishes on $A$ and $A$ is central. Also, $\nabla + \frac{1}{h} ad\, \gamma$ is monotone what with $\nabla$ preserving $W$-degree and $\frac{1}{h} ad\, \gamma$ pushing up the lowest $W$-degree in the support of an element by at least 1. This leaves the crucial issue of showing that the corrected graded derivation has square zero.

**Lemma 5.1** *For any $\gamma \in W \otimes \Omega^1$,*

$$(\nabla - \delta + \frac{1}{h} ad\, \gamma)^2 = \frac{1}{h} ad \left( -\delta(\gamma) + R + \nabla(\gamma) + \frac{1}{h}\gamma^2 \right)$$

*(Here $\nabla^2 = \frac{1}{h} ad\, R$ as in Corollary 3.1 .)*

**Proof:**

$$\begin{aligned}(\nabla - \delta + \frac{1}{h} ad\, \gamma)^2 = {} & \nabla^2 + \delta^2 + \frac{1}{h^2}(ad\, \gamma)^2 \\ & - \delta\nabla - \nabla\delta \\ & - \frac{1}{h}\left(\delta(ad\, \gamma) + (ad\, \gamma)\delta\right) \\ & + \frac{1}{h}\left(\nabla(ad\, \gamma) + (ad\, \gamma)\nabla\right).\end{aligned}$$

We analyze the right-hand side of the equation line by line. First, $\nabla^2 = \frac{1}{h} ad\, R$ and $\delta^2 = 0$. Since $\gamma$ has $\Omega$-degree 1, $(ad\, \gamma)^2 = ad(\gamma^2)$. It is easy to check that $\gamma^2 \in h(W \otimes \Omega^2)$ so we may rewrite the first line as

$$\frac{1}{h} ad(R + \frac{1}{h}\gamma^2).$$

The second line on the right-hand side is zero by Theorem 3.1. Direct calculations yield

$$\begin{aligned}\delta(ad\, \gamma) + (ad\, \gamma)\delta &= ad(\delta(\gamma)) \quad \text{and} \\ \nabla(ad\, \gamma) + (ad\, \gamma)\nabla &= ad(\nabla(\gamma)).\end{aligned}$$

The lemma follows. ■

The goal is to construct $\gamma \in \sum_{p\geq 3} W_p \otimes \Omega^1$ so that $-\delta(\gamma) + R + \nabla(\delta) + \frac{1}{h}\gamma^2$ is central. This requires some knowledge about $R$ which we delineate below.

**Lemma 5.2** *(i) $\delta R = 0$.*

*(ii) $\nabla R = dT(R)$, where $d$ is extended to a $k[[h]]$-linear map on $A[[h]] \otimes \Omega$.*

**Proof:** $(i)$ By definition, $\nabla^2 X = \frac{1}{h}[R, X]$ for all $X \in Der\, A$. Since $\delta\nabla + \nabla\delta = 0$ we know that $\delta\nabla^2 X = \nabla^2 \delta X$. Now $\delta X \in \Omega^1$, so $\nabla^2 \delta X \in d^2(\Omega^1)$. Consequently, $\nabla^2 \delta X = 0$. We infer that

$$\delta([R, X]) = 0 \quad \text{for } X \in Der\, A.$$

On the other hand, $\delta[R,X]=[\delta R,X]+[R,\delta X]$. Again, $\delta X$ is central, whence

$$[\delta R,X]=0 \quad \text{for all } X\in Der\,A.$$

It follows that $\delta R$ lies in the center of $W\otimes\Omega$.

But $R\in W_2\otimes\Omega^2$ implies $\delta R\in W_1\otimes\Omega^3$. We have already identified the center of $W\otimes\Omega$ as $A[[h]]\otimes\Omega$. Since $h$ has $W$-degree 2, there are no nonzero elements of the center with odd $W$-degree. Therefore $\delta R=0$.

($ii$) This identity is a variation on the associative law.

$$\nabla^2(\nabla X)=\frac{1}{h}[R,\nabla X] \quad \text{for } X\in Der\,A.$$

On the other hand, $\nabla^2(\nabla X)=\nabla^3 X=\nabla(\nabla^2 X)$, and

$$\nabla(\nabla^2 X)=\nabla(\frac{1}{h}[R,X])=\frac{1}{h}[\nabla R,X]+\frac{1}{h}[R,\nabla X].$$

It follows that $[\nabla R,X]=0$, i.e., $\nabla R$ is central. Put in a different form,

$$\nabla R=T(\nabla R).$$

Applying Theorem 3.2, we obtain $\nabla R=\nabla(T(R))$. Finally, $\nabla$ agrees with $d$ on $A[[h]]\otimes\Omega$. ■

**Theorem 5.1** *There exists a* $\gamma\in\overline{\sum}_{p\geq 3}W_p\otimes\Omega^1$ *such that*

$$\nabla-\delta+\frac{1}{h}ad\,\gamma$$

*has square zero.*

**Proof:** Inductively define $\gamma_m\in W_m\otimes\Omega^1$ by

$$\gamma_3=\tilde{\delta}R \quad \text{and}$$

$$\gamma_t=\frac{1}{h}\sum_{p+q-1=t}\tilde{\delta}(\gamma_p\gamma_q)+\tilde{\delta}\nabla\gamma_{t-1} \quad \text{for } t>3.$$

Then $\gamma=\overline{\sum}_{t\geq 3}\gamma_t$ satisfies

$$\gamma=\tilde{\delta}R+\frac{1}{h}\tilde{\delta}(\gamma^2)+\tilde{\delta}\nabla\gamma.$$

Since $\tilde{\delta}^2=0$, we trivially have

$$\tilde{\delta}\gamma=0. \tag{$*$}$$

Set $\beta=\delta\gamma+T(R)-R-\nabla\gamma-\frac{1}{h}\gamma^2$.

Since $\delta$ vanishes in $A[[h]] \otimes \Omega$, $\delta^2 = 0$, and $\delta R = 0$,

$$\begin{aligned}
\delta\beta &= -\delta\nabla\gamma - \frac{1}{h}\delta(\gamma^2) \\
&= -\delta\nabla\gamma + \frac{1}{h}[\gamma, \delta\gamma] \quad \text{because } \gamma \text{ has } \Omega\text{-degree } 1 \\
&= \nabla\delta\gamma + \frac{1}{h}[\gamma, \delta\gamma] \quad \text{by Theorem 3.1} \\
&= \nabla\delta\gamma + \frac{1}{h}[\gamma, \beta - T(R) + R + \nabla\gamma + \frac{1}{h}\gamma^2] \\
&= \nabla\delta\gamma + \frac{1}{h}[\gamma, \beta] + \frac{1}{h}[\gamma, R] + \frac{1}{h}[\gamma, \nabla\gamma].
\end{aligned}$$

We compare this formula to one for $\nabla\beta$:

$$\begin{aligned}
\nabla\beta &= \nabla\delta\gamma + dT(R) - \nabla R - \nabla^2\gamma - \frac{1}{h}\nabla(\gamma^2) \\
&= \nabla\delta\gamma - \frac{1}{h}[R, \gamma] - \frac{1}{h}\nabla(\gamma^2).
\end{aligned}$$

But $\nabla\gamma^2 = (\nabla\gamma)\gamma - \gamma(\nabla\gamma) = [\nabla\gamma, \gamma]$. Hence

$$\delta\beta = \nabla\beta + \frac{1}{h}[\gamma, \beta]. \tag{$i$}$$

In addition,

$$\begin{aligned}
\tilde{\delta}\beta &= \tilde{\delta}\delta\gamma - \tilde{\delta}(R + \nabla\gamma + \frac{1}{h}\gamma^2) \\
&= \tilde{\delta}\gamma - \gamma \quad \text{according to the defining relation for } \gamma \\
&= \delta\tilde{\delta}\gamma - \tau(\gamma) \quad \text{by the fundamental formula} \\
&= 0 - 0 \quad \text{by } (*).
\end{aligned}$$

Summarizing,

$$\tilde{\delta}\beta = 0. \tag{$ii$}$$

We have two-thirds of the hypotheses for the Vanishing Theorem. Obviously $\tau(\beta) = 0$ because $\beta$ has $\Omega$-degree 2. Finally, $\nabla + \frac{1}{h} ad\, \gamma$ is monotone. We conclude that $\beta = 0$. In other words,

$$-\delta\gamma + R + \nabla\gamma + \frac{1}{h}\gamma^2$$

is the central element $T(R)$. ■

**Corollary 5.1** *Fedosov connections exist.* ■

When we put together Corollaries 4.1 and 5.1 we arrive at the conclusion that any regular affine domain which is symplectic has a naive quantization.

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